Lectures on SSB in QFT

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Lecture: QFT

The Higgs mechanism

The Higgs mechanism: Generalities

The Higgs mechanism describes the spontaneous symmetry breaking of a local symmetry. Let us introduce the Higgs mechanism with an example: scalar QED; with the Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + |D_\mu \phi|^2 - V(\phi)$$

where

$$V(\phi) = -\mu^2 \phi^* \phi + \frac{\lambda}{2} (\phi^* \phi)^2$$

and

$$D_\mu = \partial_\mu + ieA_\mu$$

The Higgs mechanism: Generalities

The transformation of the fields under $U(1)$ are

$$\begin{array}{lcl} \phi(x) & \rightarrow & e^{i\alpha(x)}\,\phi(x)\,, \\ A_\mu & \rightarrow & A_\mu - \frac{1}{e}\partial_\mu \alpha(x) \end{array}$$

We find the ground state by minimizing the potential,

$$\frac{\partial V}{\partial \phi} = 0 \quad \Longleftrightarrow \quad \langle \phi \rangle = \phi_0 = \frac{\mu}{\sqrt{\lambda}} \equiv v$$

Let us change coordinates to describe excitations around this vacuum:

$$\phi(x) = \left(v + \frac{\sigma(x)}{\sqrt{2}} \right) \exp\left(\frac{i\pi(x)}{v\sqrt{2}} \right)$$

The Higgs mechanism: Generalities

The potential depends on only one of the new fields (the radial component):

$$V(\phi) = V(\sigma) = \frac{\lambda}{2}\left[\left(v + \frac{\sigma}{\sqrt{2}}\right)^2 - 2v^2\right]^2 - \frac{\lambda}{4}v^4 = \frac{1}{2}m_\sigma^2\sigma^2 + \cdots$$

with $m_\sigma^2 = 2\lambda v^2$. The covariant derivative with the new fields reads

$$D_\mu\phi = \left[\frac{\partial_\mu\sigma}{\sqrt{2}} + i\left(\frac{\partial_\mu\pi}{v\sqrt{2}} + eA_\mu\right)\left(v + \frac{\sigma}{\sqrt{2}}\right)\right]e^{\frac{i\pi(x)}{v\sqrt{2}}}$$

that gives the kinetic term

$$|D_\mu\phi|^2 = \frac{1}{2}(\partial_\mu\sigma)^2 + \left(\frac{\partial_\mu\pi}{v\sqrt{2}} + eA_\mu\right)^2\left(v + \frac{\sigma}{\sqrt{2}}\right)^2$$

$$= \frac{1}{2}(\partial_\mu\sigma)^2 + \frac{1}{2}(\partial_\mu\pi)^2 + \frac{1}{2}m_A^2A_\mu^2 + m_A(\partial_\mu\pi)A_\mu + \Delta\mathcal{L}_{\text{int}}$$

The Higgs mechanism: Generalities

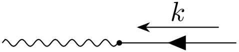
The full Lagrangian is then

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(\partial_\mu\sigma)^2 + \frac{1}{2}(\partial_\mu\pi)^2$$

$$-\frac{1}{2}m_\sigma^2\sigma^2 + \frac{1}{2}m_A^2A_\mu^2 + m_A(\partial_\mu\pi)A_\mu$$

$$+\Delta\mathcal{L}_{\text{int}}$$

We have generated an effective mass term for the photon field A_μ: the gauge bosons have acquired a mass $m_A = \sqrt{2}ev$.
However, this mass term does not come alone: it comes together with a bilinear mixing between A_μ and the Goldstone boson π. The mixing is of the form



$$= i\sqrt{2}ev(-ik^\mu) = m_A k^\mu$$

The Higgs mechanism: Generalities

Let's analyze how these two new terms contribute to modify the tree-level propagator of A_μ. Beside the $O(e^0)$ term from the Maxwell Lagrangian, we now have two $O(e^2)$ corrections connected to the two new quadratic terms

$$= i m_A^2 g^{\mu\nu} + (m_A k^\mu)\frac{i}{k^2}(-m_A k^\nu)$$
$$= i m_A^2 \left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right)$$

While a generic mass term for A_μ would have implied a complete breaking of gauge invariance and non-transverse photon propagator, here the combined action of the effective mass term and the coupling to the Goldstone boson ensures the transverse structure of the gauge boson propagator.

Generalization to non-Abelian theories

This signals that the theory is still gauge invariant. The gauge symmetry has been broken spontaneously by a specific choice of the gauge for the ground state (vacuum) of the theory.

Let us start from a very general theory with a non-Abelian gauge symmetry group G and with a system of scalar fields ϕ_i. The Lagrangian is invariant under the symmetry that transform the fields as

$$\phi_i \to (1 + i\epsilon^a t^a)_{ij}\, \phi_j \,, \qquad \delta\phi_i = i\epsilon^a t^a_{ij}\phi_j$$

we can promote it to a gauge symmetry by inserting the covariant derivative

$$D_\mu \phi = (\partial_\mu - igA^a_\mu t^a)\phi$$

Generalization to non-Abelian theories

Let's assume that

$$\left.\frac{\partial V_{\text{eff}}(\phi)}{\partial \phi_i}\right|_{\phi_i=\phi_i^0} = 0$$

yields a ground state ϕ_i^0 so that $(t^a\phi^0) \neq 0$ (for some t^a). Then $|D\phi|^2$ term at the ground state will generate

$$(D_\mu\phi)^\dagger(D^\mu\phi) \xrightarrow{\text{ground state}} (-igA_\mu^a t^a \phi^0)^\dagger(-igA_\mu^b t^b \phi^0) = \frac{1}{2}(m_{ab})^2 A_\mu^a A_\mu^b$$

with

$$m_{ab}^2 = 2g^2(t^a\phi^0)^\dagger(t^b\phi^0) = 2g^2\phi_0^\dagger(t^a t^b)\phi_0$$

(we have used $(t^a)^\dagger = t^a$).

Generalization to non-Abelian theories

The mass matrix is positive semidefinite. In fact, any diagonal element, in any basis (including the eigenvalues), has the form

$$m_{aa}^2 = 2g^2|t^a\phi^0|^2 \geq 0 \qquad \text{(no sum over a)}$$

where it is explicit that the non-vanishing masses correspond to $(t^a\phi^0) \neq 0$.

As can easily be understood, the situation is perfectly analog to the Goldstone theorem, although we deduce opposite conclusions concerning the existence of massless states: all the gauge bosons associated to broken generators ($(t^a\phi^0) \neq 0$) acquire a mass, while all the gauge bosons associated to generators that leave the vacuum invariant ($(t^a\phi^0) = 0$) remain massless.

Generalization to non-Abelian theories

In other words, there is a one-to-one correspondence between
i) the *massive gauge bosons* of a spontaneously broken *local* symmetry $G \to H$ and
ii) the *massless Goldstone bosons* of a spontaneously broken *global* symmetry $G \to H$.

In both cases these are associated to the broken generators which span the G/H coset space.

Let's present two examples.

1. Consider $SU(2)$ symmetry in scalar theory. The scalar field transforms as

$$\phi \to e^{i\alpha^a t^a}\phi$$

$t^a = \sigma^a/2$.

Generalization to non-Abelian theories

We choose the vacuum configuration as

$$\phi_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}$$

hence the mass matrix becomes

$$m_{ab}^2 = g^2 \begin{pmatrix} 0 & v \end{pmatrix} \frac{\sigma^a}{2} \frac{\sigma^b}{2} \begin{pmatrix} 0 \\ v \end{pmatrix}$$

Symmetrizing and using $1/2\{\sigma^a, \sigma^b\} = \delta^{ab}$ we get

$$m_{ab}^2 = \frac{1}{4} g^2 v^2 \delta^{ab}$$

We find three massive eigenstates of equal mass. This will be useful for the SM where we also have 3 massive vectors.

Generalization to non-Abelian theories

2. Consider now a real field $\phi = \{\phi_1, \phi_2, \phi_3\}$ transforming into the adjoint representation of $SU(2)$. The covariant derivative is

$$D_\mu \phi^a = \partial_\mu \phi^a + ig\epsilon^{abc} A_\mu^b \phi^c$$

Choosing the vacuum configuration

$$\phi_0 = \{0, 0, v\}$$

we obtain

$$\frac{1}{2}(D_\mu \phi^a)(D^\mu \phi^a) \to \frac{g^2}{2}(\epsilon_{ab3} A_\mu^b)(\epsilon_{ac3} A_\mu^c) v^2 = \frac{g^2}{2} v^2 \left[(A_\mu^1)^2 + (A_\mu^2)^2 \right]$$

where in the last equality we used the identity

$$\sum_a \epsilon_{ab3}\epsilon_{ac3} = \delta_{b1}\delta_{c1} + \delta_{b2}\delta_{c2}$$

Generalization to non-Abelian theories

In this case we obtain two massive eigenstates, i.e., $m_1^2 = m_2^2 = g^2 v^2$ while $m_3^2 = 0$ since there is a residual $U(1)$ symmetry associated with t^3.

Another way to see this is by representing the scalar filed in the adjoint as a 2×2 real and traceless matrix (which has indeed three independent components).

In this alternative (but equivalent) notation, the field transforms as

$$\delta\phi_{ij} = \epsilon^a [t^a, \phi]_{ij}$$

and the covariant derivative reads

$$D_\mu \phi_{ij} = \partial_\mu \phi_{ij} + i[t^a, \phi]_{ij} A_\mu^a$$

Generalization to non-Abelian theories

The chosen vacuum

$$\phi_0 = \{0, 0, v\}$$

is equivalent to

$$\phi_0 \propto \begin{pmatrix} v & 0 \\ 0 & -v \end{pmatrix}$$

and commute with $t^3 = \sigma^3/2$ but not with σ^1 and σ^2.

Since the vacuum breaks 2 symmetries, two gauge boson acquire a non-vanishing mass.

Quantization of spontaneously broken gauge theories

The R_ξ gauge

Let us consider the complex scalar theory with Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + |D_\mu \phi|^2 - V(\phi)$$

with $D_\mu = \partial_\mu + ieA_\mu$ and $\phi = 1/\sqrt{2}\,(\phi_1 + i\phi_2)$. The Lagrangian is invariant under $U(1)$ symmetry, whose infinitesimal action on the fields is

$$\delta\phi_1 = -\alpha(x)\phi_2(x) \qquad \delta\phi_2 = \alpha(x)\phi_1(x) \qquad \delta A_\mu = -\frac{1}{e}\partial_\mu \alpha(x)$$

where $\alpha(x)$ is a real function. Let us assume the (effective) potential is minimized by $\langle\phi\rangle = v/\sqrt{2}$, and we will use the following vacuum configuration: $\phi_1 = v$, $\phi_2 = 0$.

The R_ξ gauge

As we have already seen, it is convenient to redefine the fields as

$$\phi_1(x) = v + h(x)\,, \qquad \phi_2 = \varphi(x)$$

such that the covariant derivative assumes the form

$$D_\mu \phi = \frac{1}{\sqrt{2}} \left[\partial_\mu h + i\partial_\mu \varphi + ieA_\mu (v+h) - eA_\mu \varphi \right]$$

The Lagrangian written in terms of the new fields is

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} (\partial_\mu h - eA_\mu \varphi)^2 + \frac{1}{2} \left(\partial_\mu \varphi + eA_\mu (v+h) \right)^2 - V(h,\varphi)$$

The R_ξ gauge

Let's proceed analyzing the quantization of this Lagrangian. As usual, we start introducing the generating functional

$$W[J] \propto \int \mathcal{D}A\,\mathcal{D}h\,\mathcal{D}\varphi\, e^{i\int \mathcal{L}[A,h,\varphi;J]}$$

Following the Faddeev-Popov gauge fixing procedure, we insert the identity

$$1 = \int \mathcal{D}\alpha\,\delta\left[G(A^\alpha)\right]\det\left[\frac{\delta G(A^\alpha)}{\delta\alpha}\right]$$

A^α denotes the gauge transformed field $A_\mu \to A_\mu^\alpha$.

We proceed with changing the variables $A \to A^\alpha$ in $W[J]$ with taking into account $\mathcal{D}A = \mathcal{D}A^\alpha$ and the gauge invariance of the Lagrangian $\mathcal{L}[A] \to \mathcal{L}[A^\alpha]$.

The R_ξ gauge

Renaming A^α to A we get

$$W[J] \propto \int \mathcal{D}\alpha\, \mathcal{D}A\, \mathcal{D}h\, \mathcal{D}\varphi\, e^{i\int \mathcal{L}[A,h,\varphi;J]}\, \delta[G(A)]\, \det\left[\frac{\delta G(A)}{\delta\alpha}\right]$$

We further choose the gauge fixing condition of the form

$$G(A) = \omega(x)$$

that is applied in the path integral with $\delta(G(A) - \omega(x))$. Then we can integrate over $\omega(x)$ with a Gaussian weight

$$\int \mathcal{D}\omega\, e^{-\frac{i}{2}\omega^2(x)} \delta[G(A) - \omega(x)]$$

to obtain

$$W[J] \propto \int \mathcal{D}\alpha\, \mathcal{D}A\, \mathcal{D}h\, \mathcal{D}\varphi\, e^{i\int\left[\mathcal{L} - \frac{1}{2}G(A)^2\right]}\, \det\left[\frac{\delta G(A)}{\delta\alpha}\right]$$

The R_ξ gauge

Before, we proceeded with $G(A) = \partial_\mu A^\mu$. However, in the present case it is a smart choice to be

$$G(A) = \frac{1}{\sqrt{\xi}}(\partial_\mu A^\mu - \xi e v \varphi)$$

As we shall see, this term has the advantage of eliminating the mixed term $A^\mu \partial_\mu \phi$ in the quadratic part of the complete Lagrangian. This part has the form

$$\mathcal{L}_2 = -\frac{1}{2}(\partial_\mu A_\nu)^2 + \frac{1}{2}\partial_\mu A_\nu \partial^\nu A^\mu + \frac{1}{2}(\partial_\mu \varphi)^2 + \frac{1}{2}e^2 v^2 A_\mu^2 + \underline{ev\partial_\mu \varphi A_\mu}$$

$$+ \frac{1}{2}(\partial_\mu h)^2 - \frac{1}{2}m_h^2 h^2$$

The R_ξ gauge

Since
$$-\frac{1}{2}G^2 = -\frac{1}{2\xi}(\partial_\mu A^\mu)^2 + \underline{ev\partial_\mu A^\mu \varphi} - \frac{\xi}{2}e^2v^2\varphi^2$$

the underlined terms cancel each other after integration by parts in the exponential term of $W[J]$.

The quadratic part of the effective Lagrangian after gauge fixing is then

$$\mathcal{L}_2 - \frac{1}{2}G^2 = -\frac{1}{2}A_\mu\left[-g^{\mu\nu}\partial^2 + \left(1-\frac{1}{\xi}\right)\partial^\mu\partial^\nu - (ev)^2 g^{\mu\nu}\right]A_\nu$$
$$+\frac{1}{2}(\partial_\mu\varphi)^2 - \frac{\xi}{2}(ev)^2\varphi^2 + \frac{1}{2}(\partial_\mu h)^2 - \frac{1}{2}m_h^2 h^2$$

We have generated ξ-dependent unphysical mass of the Goldstone boson φ

$$m_\varphi^2 = \xi m_A^2 \qquad \text{with} \qquad m_A^2 = e^2v^2$$

The mass is gauge-dependent hence the Goldstone boson is a fictitious field, which cannot be produced in physical processes.

The R_ξ gauge

With the choice of $G(A)$ we have ghosts in our theory. (Though we are in Abelian theory, there still are ghosts!)

From the infinitesimal transformations $\delta A_\mu = -1/e\,\partial_\mu \alpha$, $\delta\varphi = \alpha\phi_1 = \alpha(v + h(x))$ it follows that

$$\frac{\delta G}{\delta\alpha} = \frac{1}{\sqrt{\xi}}\left[-\frac{1}{e}\partial^2 - \xi e v(v+h)\right]$$

with a non-trivial dependence on $h(x)$.
The functional determinant can be accounted for in the Lagrangian by introducing a pair of ghost fields. Re-absorbing an overall factor $1/e$ we can write describe the functional determinant via

$$\mathcal{L}_{\text{ghost}} = \bar{c}\left[-\partial^2 - \xi m_A^2\left(1+\frac{h}{v}\right)\right]c$$

The R_ξ gauge

From $\mathcal{L}_{\text{ghost}}$ we have ξ-dependent mass for ghost fields $m^2_{\text{ghost}} = \xi m_A^2$.

Note that, contrary to the non-Abelian case without spontaneous symmetry breaking, here the ghost fields only couple to the massive scalar field (the Higgs field) and not to the gauge boson(s).

As a last step, we can finally derive the propagators of the various fields appearing in the theory. The quadratic part of the Lagrangian involving A_μ is

$$\mathcal{L}_{2A} = -\frac{1}{2} A_\mu \left[g^{\mu\nu}(-\partial^2 - m_A^2) + \left(1 - \frac{1}{\xi}\right) \partial^\mu \partial^\nu \right] A_\nu = -\frac{1}{2} A_\mu \hat{K}^{\mu\nu} A_\nu$$

In the momentum space, $\tilde{K}^{\mu\nu}$ assumes to have the form

$$g^{\mu\nu}(k^2 - m_A^2) - \left(1 - \frac{1}{\xi}\right) k^\mu k^\nu$$

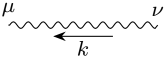
$$= \left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2} \right) (k^2 - m_A^2) + \frac{k^\mu k^\nu}{k^2} \frac{1}{\xi} (k^2 - \xi m_A^2)$$

The R_ξ gauge

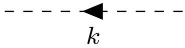
It implies the following structure for the propagator of A_μ in momentum space:

$$\langle A^\mu(k)A^\nu(-k)\rangle = (\hat{K}^{\mu\nu})^{-1} = \frac{-i}{k^2-m_A^2}\left(g^{\mu\nu}-\frac{k^\mu k^\nu}{k^2}\right)+\frac{-i\xi}{k^2-\xi m_A^2}\frac{k^\mu k^\nu}{k^2}$$
$$= \frac{-i}{k^2-m_A^2}\left(g^{\mu\nu}-(1-\xi)\frac{k^\mu k^\nu}{k^2-\xi m_A^2}\right).$$

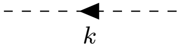
The other propagators can be derived in a straightforward way, and the full list of propagators comes as follows:

A_μ : 

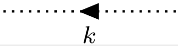
$$= \frac{-i}{k^2-m_A^2}\left(g^{\mu\nu}-\frac{k^\mu k^\nu}{k^2-\xi m_A^2}(1-\xi)\right)$$

h : 

$$= \frac{i}{k^2-m_h^2}$$

φ : 

$$= \frac{i}{k^2-\xi m_A^2}$$

c : 

$$= \frac{i}{k^2-\xi m_A^2}$$

The R_ξ gauge

As can be seen, the transverse components of the gauge field, and the massive scalar h (the Higgs particle) acquire gauge-independent (physical) masses: m_A and m_h. On the other hand, the unphysical components of A, the Goldstone bosons and the ghosts all acquire the same gauge-dependent (unphysical) mass $m_\varphi = \sqrt{\xi} m_A$.

Three notable choices for ξ are worth to be discussed

- **Coulomb gauge** $\xi = 0$. In this case the Goldstone bosons are massless (as in in the case of a global symmetry) and A has a transverse propagator.
- **Feynman gauge** $\xi = 1$. In this case $m_\varphi = m_A$, and the propagator of A has no terms proportional to $k^\mu k^\nu$.

The R_ξ gauge

- **Unitary gauge** $\xi = \infty$. In this case the Goldstone bosons and ghosts decouple, acquiring an infinite mass ($m_\varphi \to \infty$). The gauge field has the propagator expected for a massive vector field with 3 independent polarizations:

$$\langle A^\mu(k) A^\nu(-k) \rangle|_{\text{unitarygauge}} = \frac{-i}{k^2 - m_A^2} \left(g^{\mu\nu} - \frac{k^\mu k^\nu}{m_A^2} \right)$$

In the unitary gauge, the Goldstone boson is "absorbed" into the longitudinal polarization of the gauge boson, that acquire mass: all the degrees of freedom of the theory are described by the massive gauge field.

This gauge, which is quite useful to understand the spectrum of the theory, should be used with care when doing loop calculations since it corresponds to a singular limit of the Lagrangian.

The R_ξ gauge

In any other gauge (finite ξ) we have a "constrained" vector propagator, reflecting the two physical polarizations of the gauge field in absence of spontaneous symmetry breaking (this is particularly clear in the Coulomb gauge) plus a Goldstone boson. In all cases the effective number of degrees of freedom is conserved.

In other words, there is unavoidable gauge-dependent mixing between the longitudinal polarization of the gauge boson and the Goldstone boson: in physical processes, the combined effect due to the propagation of these two modes, which are separately gauge-dependent, leads to the restoration of gauge invariance.

Note also that the mass of the gauge field, $m_A \neq 0$, is a true physical effect: there is no physical pole at $k^2 = 0$ in the S matrix.

A simplified model with chiral symmetry breaking

To better understand the interplay between the longitudinal polarization of the gauge bosons and the Goldstone bosons, it is worth to analyze a slightly more sophisticated model, which is also a good prototype to understand the structure of weak interactions in the Standard Model.

Let us start again from a theory with $U(1)$ local symmetry and a scalar field. In addition to a complex scalar field ϕ we add a left-handed ψ_L (charged under $U(1)$ with the same charge as ϕ), and a right-handed ψ_R uncharged under $U(1)$.

Also we add a Yukawa type interaction between the scalar and the fermions. Then the fermionic part of the action is

$$\Delta\mathcal{L}_f = \bar{\psi}_L(i\not{D})\psi_L + \bar{\psi}_R(i\not{\partial})\psi_R - \lambda_f(\bar{\psi}_L\phi\psi_R + \bar{\psi}_R\phi^*\psi_L)$$

Here $D_\mu = \partial_\mu + ieA_\mu$.

A simplified model with chiral symmetry breaking

The gauge transformations of scalar fermion fields are

$$\psi_L \to e^{i\alpha(x)}\psi_L \qquad \phi \to e^{i\alpha(x)}\phi \qquad \psi_R \to \psi_R$$

According to transformations ψ_L and ψ_R are independent fields (2-spinors for each), with different properties upon the Lorentz transformations and different gauge.

After spontaneous symmetry breaking, they can be viewed as the two chiralities of a unique (4-component) massive Dirac fermion. Indeed, writing

$$\psi_L = P_L\psi \ , \qquad \psi_R = P_R\psi \ , \qquad P_{L,R} = \left(\frac{1 \mp \gamma_5}{2}\right)$$

we see the appearance the Dirac mass term $m_f\bar{\psi}\psi$ after the spontaneous symmetry breaking and setting the V.E.V. of ϕ.

A simplified model with chiral symmetry breaking

The mass m_f becomes
$$m_f = \lambda_f \frac{v}{\sqrt{2}}$$

Also the gauge field A_μ acquire a mass $m_A = ev$, and the coupling of the fermion to the gauge field is

$$\Delta\mathcal{L}_{\psi-A} = -e\bar{\psi}_L\gamma^\mu\psi_L A_\mu = -e\bar{\psi}\gamma^\mu P_L\psi A_\mu$$

The couplings of the Goldstone boson φ and the massive scalar h to the fermions ψ are

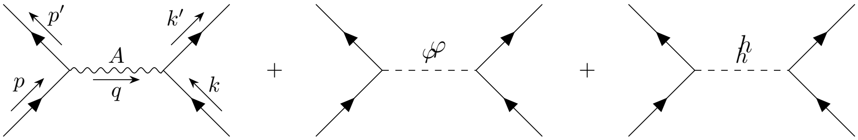
$$\Delta\mathcal{L}_{\varphi/h-\psi} = -\frac{\lambda_f}{\sqrt{2}}[\bar{\psi}_L(\phi_1 + i\phi_2)\psi_R + \bar{\psi}_R(\phi_1 - i\phi_2)\psi_L]$$

$$= -\frac{\lambda_f}{\sqrt{2}}[(\bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L)(v+h) + i(\bar{\psi}_L\psi_R - \bar{\psi}_R\psi_L)\varphi]$$

$$= -\frac{\lambda_f}{\sqrt{2}}[\bar{\psi}\psi(v+h) + i\bar{\psi}\gamma_5\psi\varphi]$$

A simplified model with chiral symmetry breaking

We are now interested to compute the scattering $\psi\psi \to \psi\psi$ in this theory. We will do the calculation a the tree level, in a generic R_ξ gauge. The ghosts do not appear in this process until the one-loop level and the only contributing diagrams are



One may analyze diagrams to make the following conclusions.

A simplified model with chiral symmetry breaking

So, analysing each diagram (starting from the right), we deduce that

- The h-boson exchange diagram is ξ-independent. As a result, the ξ-dependence must cancel between the Goldstone-boson and the gauge-boson exchange diagrams.
- The Goldstone boson exchange diagrams generates the following contribution to the amplitude

$$i\mathcal{M}_\varphi = \left(\frac{\lambda_f}{\sqrt{2}}\right)^2 \bar{u}(p')\gamma^5 u(p) \frac{i}{q^2 - \xi m_A^2} \bar{u}(k')\gamma^5 u(k)$$

- The gauge boson exchange diagram leads to

$$i\mathcal{M}_A = (-ie)^2 \bar{u}(p')\gamma_\mu P_L u(p) \frac{-i}{q^2 - m_A^2} \left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2 - \xi m_A^2}(1-\xi)\right) \bar{u}(k')\gamma_\nu P_L u(k)$$

A simplified model with chiral symmetry breaking

We can rewrite the gauge boson propagator as

$$\frac{-i}{q^2 - m_A^2}\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{m_A^2} + q^\mu q^\nu\left[\frac{1}{m_A^2} - \frac{1}{q^2 - \xi m_A^2}(1-\xi)\right]\right)$$

$$= \frac{-i}{q^2 - m_A^2}\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{m_A^2}\right) + \frac{-i}{q^2 - \xi m_A^2}\left(\frac{q^\mu q^\nu}{m_A^2}\right)$$

The first term is ξ independent, and is the result we would have obtained in the unitary gauge. The second term can be simplified after contracting the spinors

$$\begin{aligned} q^\mu \bar{u}(p')\gamma_\mu P_L u(p) &= \frac{1}{2}\bar{u}(p')[(\not{p} - \not{p}') - (\not{p} - \not{p}')\gamma^5]u(p) \\ &= \frac{1}{2}\bar{u}(p')[\not{p}'\gamma^5 + \gamma^5\not{p}]u(p) = m_f\bar{u}(p')\gamma^5 u(p) \end{aligned}$$

$q = p - p' = k - k'$ is the momentum transfer.

A simplified model with chiral symmetry breaking

The explicit expressions for m_A and m_f we find that the term depending on ξ in $i\mathcal{M}_A$ is

$$\left(\frac{\lambda_f}{\sqrt{2}}\right)^2 \bar{u}(p')\gamma^5 u(p) \frac{-i}{q^2 - \xi m_A^2} \bar{u}(k')\gamma^5 u(k)$$

that precisely cancels the Goldstone boson exchange diagram.

This result nicely illustrate the role of the Goldstone boson in ensuring a gauge-invariant result.

The Goldstone boson equivalence theorem

The results we got in before can easily be generalized to the case of non-Abelian gauge symmetries: the Goldstone bosons associated to the broken generators are "eaten" by the corresponding gauge field, that acquire a non-vanishing mass.

What we will show in this section is that, despite we can get rid of the Goldstone bosons in the unitary gauge, at high energies (i.e. at energies well above the masses of the gauge fields), the amplitude for emission or absorption of a longitudinally polarized massive gauge boson becomes equal to the amplitude for the emission or absorption of the corresponding Goldstone boson.

In other words, at high energies we restore a system where, for each broken generator, two degrees of freedom are associated to the transverse gauge boson and the third one is associated to the the Goldstone boson field.

The Goldstone boson equivalence theorem

To derive this result, known as the Goldstone boson equivalence theorem, it is worth to revisit the Goldstone boson theorem and the Higgs mechanism using a different (more general) language.

Let us first consider a Lagrangian $\mathcal{L}_0$ with a global non-abelian symmetry G. The infinitesimal transformation of $\mathcal{L}_0$ under G is

$$\delta_G \mathcal{L}_0 = \partial_\mu \alpha^a \, J_\mu^a$$

that leads to the conservation of the current $\partial_\mu J_a^\mu = 0$.
If we promote G to become a local symmetry, adding an appropriate set of gauge bosons (A_μ^a), the Lagrangian of the theory is modified as follows

$$\mathcal{L}_0 \to \mathcal{L} = \mathcal{L}_0 - g A_\mu^a J_\mu^a \quad + \quad \underbrace{O(A^2)}_{\text{kinetic terms, } \ldots}$$

The Goldstone boson equivalence theorem

Because of the Ward Identity of the global symmetry, we have

$$\partial_\mu \langle f| J^a_\mu |i\rangle = 0$$

$|i\rangle$ and $|j\rangle$ are physical on-shell states.

If the symmetry is spontaneously broken, that is $T^a|0\rangle \neq 0$ for some generators T^a of group G, the current J^a_μ has the right quantum numbers to create Goldstone bosons $|\pi_k\rangle$ from the vacuum state.

Lorentz invariance implies the following general decomposition of the $\langle 0|J^a_\mu|\pi_k\rangle$ matrix element

$$\langle 0| J^a_\mu(x) |\pi_k(p)\rangle = -ip_\mu F^a_k e^{-ip\cdot x}$$

By construction, F^a_k can be non vanishing only if $T^a|0\rangle \neq 0$, hence a non-vanishing F^a_k implies that we have a spontaneous symmetry breaking.

The Goldstone boson equivalence theorem

To better understand the last formula, we note following two observations:

- The current conservation implies

$$0 = \partial_\mu \langle 0 | J^a_\mu(x) | \pi_k(p) \rangle = -ip^2 F^a_k e^{-ipx}$$

  Hence we deduce that for an on-shell $\pi_k(p)$ state we must have $p^2 = 0$, which is exactly what we expect for a Goldstone boson (i.e. π_k is a massless state).

- In a scalar theory with $SU(N)$ symmetry, where a scalar field ϕ in a given representation acquires a vacuum expectation value $\phi^i_0 = (\phi^i_0)^\dagger$, we have

$$D_\mu \phi^\dagger D^\mu \phi \supset -g A^a_\mu (\sqrt{2} T^a_{kj} \phi^0_j) \frac{(i\partial^\mu \phi^*_k - i\partial^\mu \phi_k)}{\sqrt{2}} = -g A^a_\mu (\sqrt{2} T^a_{ij} \phi^0_j)(\partial^\mu \pi_k)$$

The Goldstone boson equivalence theorem

In the latter result we have used $\delta\phi_i = i\epsilon^a T^a_{ij}\phi_j$ and write down the scalar field as $\phi_i = \phi_i^0 + 1/\sqrt{2}\,(h_i + i\pi_i)$. The above result implies

$$J^a_\mu \supset (\sqrt{2}T^a_{ij}\phi^0_j)(\partial^\mu \pi_k)$$

which in turn allows us to deduce that $F^a_k = \sqrt{2}T^a_{ij}\phi^0_j$.

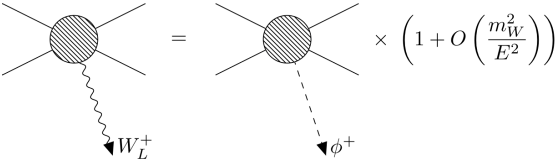
Therefore, as expected $F^a_k \neq 0$ if $T^a_{ij}\phi^0_j \neq 0$.

Note also that the gauge boson mass matrix can be written as $m^2_{ab} = g^2 F^a_k F^b_k$, a result that holds independently of the specific mechanism of spontaneous symmetry breaking.

The Goldstone boson equivalence theorem

The Goldstone boson equivalence theorem

Now we can prove (a simplified version) of the equivalence theorem between Goldstone bosons and longitudinal components of the gauge bosons. The main idea can be naively illustrated as follows



At high energies the amplitude for emission or absorption of a longitudinally polarized massive gauge boson becomes equal to the equivalent amplitude with the gauge boson replaced by the corresponding "eaten" Goldstone boson.

Denoting W_μ the massive gauge boson, let us analyze its polarization vectors.

The Goldstone boson equivalence theorem

W **at rest:** $k^\mu = (m, \vec{0})$
By construction, in order to satisfy the conditions

$$\begin{cases} \epsilon_\mu k^\mu = 0 \\ \epsilon^2 = -1 \end{cases}$$

a polarization vector must be a linear combination of the three orthogonal unit vectors

$$\epsilon_1 = (0, 1, 0, 0) \ , \quad \epsilon_2 = (0, 0, 1, 0) \ , \quad \epsilon_3 = (0, 0, 0, 1)$$

The Goldstone boson equivalence theorem

W **boosted along the 3rd spatial direction:** $k_\mu = (E_k, 0, 0, |\vec{k}|)$.
In order to satisfy,

$$\begin{cases} \epsilon_\mu k^\mu = 0 \\ \epsilon^2 = -1 \end{cases}$$

the transverse polarizations are given by ϵ_1 and ϵ_2, while the longitudinal polarization vector is given by

$$\epsilon_L^\mu(k) = \left(\frac{|\vec{k}|}{m}, \quad 0, \quad 0, \quad \frac{E_k}{m} \right)$$

At large momentum ($E_k \sim |\vec{k}| \gg m$)

$$\epsilon_L^\mu(k) \sim \frac{k^\mu}{m}$$

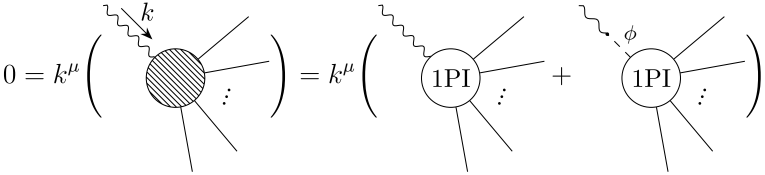
that is ϵ_L^μ becomes $\|$ to k^μ.

The Goldstone boson equivalence theorem

Now consider a generic matrix element of the conserved current J^a_μ, and impose the current conservation in momentum space:

$$k^\mu \langle f | \, J^a_\mu \, | i \rangle = 0$$

Diagrammatically, it means



The Goldstone boson equivalence theorem

More explicitly,

$$k^{\mu}\left[\Gamma_{\mu}^{(W)}(k)+igFk_{\mu}\frac{i}{k^{2}}\Gamma^{(\mathrm{GB})}(k)\right]=0$$

$\Gamma_{\mu}^{(W)}(k)$ denotes the one-particle irreducible (1PI) with the emission of a massive gauge boson of momentum k; $\Gamma^{(\mathrm{GB})}(k)$ the one corresponding Goldstone boson. Since $m_W = gF$, we deduce that

$$k^{\mu}\Gamma_{\mu}^{(W)}(k)=m_{W}\Gamma^{(\mathrm{GB})}(k)$$

In the high-energy limit we get

$$\epsilon_{L}^{\mu}(k)\Gamma_{\mu}^{(W)}(k)=\Gamma^{(\mathrm{GB})}(k)$$

and it completes the proof.

The Goldstone boson equivalence theorem

Namely, at high energies, the contribution to physical amplitudes of the longitudinal polarization of a massive gauge boson, from a spontaneously broken gauge symmetry, is equivalent to that of the corresponding Goldstone boson.

Let's consider the following example: **the top quark decay in the Standard Model**.

Within the Standard Model (SM), quark masses, as well as the masses of the weak gauge bosons, are the result of the spontaneous symmetry breaking of a (complex) scalar field H (the Higgs field) transforming as a doublet of the $SU(2)_L$ gauge symmetry.

The Goldstone boson equivalence theorem

The key ingredients to describe the decay of the t quark into a W and a b quark are the following terms in the Lagrangian

$$\Delta\mathcal{L}_{\rm top-Yukawa} \;=\; y_t\bar{q}_L(i\sigma_2 H^*)t_R + \text{h.c} \quad \xrightarrow{\langle H\rangle} \quad \frac{y_t v}{\sqrt{2}}\,(\bar{t}_L t_R + \text{h.c.}) + \ldots$$

$$\Delta\mathcal{L}_{\rm t-W} \;=\; \bar{q}_L(i\not{D})q_L \supset \bar{t}_L\gamma^\mu b_L W^+_\mu + \text{h.c.}, \qquad q_L = \begin{pmatrix} t_L \\ b_L \end{pmatrix}$$

$$\Delta\mathcal{L}_{\rm W-mass} \;=\; (D_\mu H)^\dagger D_\mu H \quad \xrightarrow{\langle H\rangle} \quad \frac{g^2v^2}{4}\,W^-_\mu W^+_\mu + \ldots$$

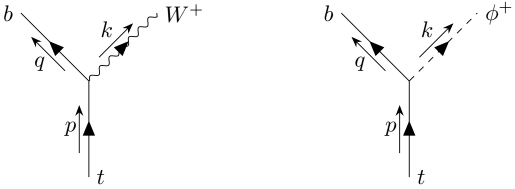
The terms indicated with an arrow are those obtained replacing the Higgs field with its vacuum expectation value

$$\langle 0|H|0\rangle = \frac{v}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The Goldstone boson equivalence theorem

From those we deduce that $m_t = y_t v/\sqrt{2}$ and $m_W^2 = g^2 v^2/4$.

The tree-level diagram describing the decay of the t quark into a W^+ and a b quark (which we can treat as massless, in first approximation), is described by the following diagrams



Computing this diagram in the unitary gauge we obtain the complete result at $O(g)$. The corresponding amplitude is

$$i\mathcal{M} = \frac{ig}{\sqrt{2}}\bar{u}(q)\gamma^\mu P_L u(p)\epsilon_\mu^*(k)$$

The Goldstone boson equivalence theorem

Squaring $\mathcal{M}$, averaging over all the possible W polarizations, and summing over final spins, leads to

$$\begin{aligned}
\frac{1}{2}\sum_{\text{spins}}|\mathcal{M}|^2 &= \frac{g^2}{2}\mathrm{Tr}\left[u(q)\bar{u}(q)\gamma^\mu u(p)\bar{u}(p)\gamma^\nu\right]\sum_{\text{polariz.}}\epsilon^*_\mu(k)\epsilon_\nu(k) \\
&= \frac{g^2}{2}[q^\mu p^\nu + q^\nu p^\mu - g^{\mu\nu} q\cdot p]\left[-g_{\mu\nu} + \frac{k_\mu k_\nu}{m_W^2}\right] \\
&= \frac{g^2}{2}\left[q\cdot p + 2\frac{(k\cdot q)(k\cdot p)}{m_W^2}\right]
\end{aligned}$$

The Goldstone boson equivalence theorem

Taking into account the decay kinematics, this leads to

$$\frac{1}{2}\sum_{\text{spins}}|\mathcal{M}|^2 = g^2\frac{m_t^4}{4m_W^2}\left[1+O\left(\frac{m_W^2}{m_t^2}\right)\right]$$

that implies

$$\Gamma(t\to bW^+)_{\text{SM}} = \frac{g^2}{64\pi}\frac{m_t^3}{m_W^2}\left[1+O\left(\frac{m_W^2}{m_t^2}\right)\right]$$

Naively, this result seems to imply that the decay width of the t quark diverges in the limit $m_W \to 0$; however, this is clearly an artefact since $m_W \propto g$.

The Goldstone boson equivalence theorem

To better understand what happens in the limit $m_t \gg m_W$ we can make use of the Goldstone boson equivalence theorem. In this limit the W is highly boosted and the contribution to the amplitude is dominated by the decay into a longitudinally polarized W, with amplitude

$$\begin{aligned} i\mathcal{M}_L &= \frac{ig}{\sqrt{2}}\bar{u}(q)\gamma^\mu P_L u(p)\frac{k^\mu}{m_W} = \frac{ig}{\sqrt{2}}\bar{u}(q)(\not{p} - \not{q})P_L u(p)\frac{1}{m_W} \\ &= \frac{ig}{\sqrt{2}}\frac{m_t}{m_W}\,\bar{u}(q)P_L u(p) = i\left(\frac{\sqrt{2}m_t}{v}\right)\bar{u}(q)P_L u(p) \end{aligned}$$

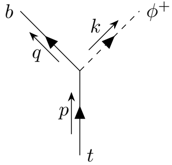
Since $y_t = \sqrt{2}m_t/v$ it's easy to realize

$$i\mathcal{M}_L = iy_t\bar{u}(q)P_L u(p) = i\mathcal{M}_{GB}$$

$\mathcal{M}_{GB}$ is the amplitude for the decay to a Goldstone boson, in absence of gauging of the symmetry, as expected by the Goldstone boson equivalence theorem.

The Goldstone boson equivalence theorem

In this specific process we can understand this phenomenon also noting that $m_t \gg m_W$ necessarily imply $y_t \gg g$. Hence we can "switch-off" gauge interactions and analyze the process in the limit $g \to 0$. In such limit there is no W, and the t quark decay via $t \to b^+$, corresponding to the diagram



The corresponding decay width is

$$\Gamma(t \to bW^+)_0 = y_t^2 \frac{m_t}{32\pi}$$

and this is the leading term in the previously obtained decay when $m_t \gg m_W$. (No g and m_W dependence!)