Методи квантової теорії поля у фізиці частинок
Лекція: симетрії у фізиці частинок

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Симетрії у фізиці частинок і теорія груп. Симетрія поворотів, ізоспінова симетрія та симетрія ароматів

Symmetries play an important role in elementary particle physics, because of their relation to conservation laws and also because they permit one to make progress when we do not have a complete dynamical theory.

Symmetries, Groups, and Conservation Laws.

In 1917 the dynamical implications of symmetry were completely understood. In that year, Emmy Noether published her famous theorem relating symmetries and conservation laws:

$$Noether's\ Theorem: \qquad Symmetries \leftrightarrow Conservation\ laws$$

Symmetry		Conservation Law
Translation in time	$\leftrightarrow$	energy E
Translation in space	$\leftrightarrow$	3-momentum $\vec{P}$
Rotation of our 3-dimensional space	$\leftrightarrow$	angular momentum $\vec{J}$
Gauge transformations	$\leftrightarrow$	electric charge, baryon number Q, B

Every symmetry of nature yields a conservation law; conversely, every conservation law reflects an underlying symmetry.

What precisely is a symmetry? It is an operation you can perform on a system that leaves it invariant - that carries it into a configuration indistinguishable from the original one.

In mathematics the **group theory** may be regarded as the systematic study of symmetries. The symmetry operations R_i are combined in a group, if they obey the properties:

- Closure: $R_i \cdot R_j = R_k$ for any i, j and all R_i, R_j, R_k belong to this group,
- Identity: there is identity element I, such that $R_i \cdot I = I \cdot R_i = R_i$ for any R_i,
- Inverse: for any R_i there is inverse element R_i^{-1}, such that $R_i \cdot R_i^{-1} = R_i^{-1} \cdot R_i = I$,
- Associativity: $R_i \cdot (R_j \cdot R_k) = (R_i \cdot R_j) \cdot R_k$

What groups do we know?

– Abelian: $R_i \cdot R_j = R_j \cdot R_i$. Examples: translation in time, translations in space
– Nonabelian: $R_i \cdot R_j \neq R_j \cdot R_i$. Example: rotation of 3-dimensional space
– Finite group: $i = 1, 2, 3, ..., N$. Example: group of symmetry of of the equilateral triangle ($N = 6$)
– Infinite: $i = 1, 2, 3, ..., \infty$. Example: all integer numbers $\ldots -2, -1, 0, 1, 2, \ldots$ with addition of two numbers as operation $\cdot$, i.e. $(-5) \cdot 6 \equiv -5 + 6 = +1$

Symmetry in physics and groups

– Continuous: elements depend on one or more continuous parameters, e.g. translation in space, time, or rotation,
- Discrete: elements are numbered by index which takes integer values, 1,2,3, ...

In physics we are mostly interested in continuous groups of matrices.

Group	Matrices	Symmetry in physics
$U(n)$	$n \times n$ unitary: $U^\dagger U = UU^\dagger = 1$	for $n = 1$ electric charge, baryon number, lepton numbers
$SU(n)$	$n \times n$ unitary and $det(U) = 1$	for $n = 2$ isospin, for $n = 3$ flavor and color symmetry in QCD
$O(n)$	$n \times n$ orthogonal: $O^T O = OO^T = 1$	
$SO(n)$	$n \times n$ orthogonal, and $det(O) = 1$	for $n = 3$ rotations of 3-dimensional space

Here "U" means unitary, "S" means special, "O" means orthogonal. Also we have $U^\dagger = U^{T*}$, and T means transposition $U^T_{ij} = U_{ji}$. Lorentz transformations are also described by the continuous matrices.

Finally, there are **representations** of group. This means that for any element of a group R_i there is matrix M_{R_i}, and also for $R_i R_j = R_k$ there is $M_{R_i} M_{R_j} = M_{R_k}$. These matrices can have any dimension, 1×1, 2×2, etc. If we have initially group, say $SU(3)$, then representation with matrices 3×3 is called **fundamental representation**.
If you are interested in the **group theory** you could read some books in mathematics. Here we will be considering the most important groups in particle physics.

Rotation symmetry and angular momentum

Most of this you know from Quantum mechanics. Here I will briefly tell you the aspects of it related to particle physics.

In general, angular momentum $\vec{J}$ includes orbital momentum $\vec{L}$ and spin $\vec{S}$

$$\vec{J} = \vec{L} + \vec{S}$$

According to quantum theory we can measure only

$$\vec{L}^2 = l(l+1)\hbar^2, \quad \textit{with} \quad l = 0, 1, 2, \dots$$

$$L_z = m_l\hbar, \quad \textit{with} \quad m_l = -l, -l+1, \dots, l-1, l, \quad \textit{in total}\ (2l+1)\ \textit{values}$$

For the spin similarly

$$\vec{S}^2 = s(s+1)\hbar^2, \quad \textit{with} \quad s = 0, 1/2, 1, 3/2, 2, \dots$$

$$S_z = m_s\hbar, \quad \textit{with} \quad m_s = -s, -s+1, \dots, s-1, s, \quad \textit{in total}\ (2s+1)\ \textit{values}$$

The main difference between the orbital momentum and spin is that **every particle can have any orbital momentum $l = 0, 1, 2, \dots$, but it has only one fixed value of spin** s.

All particles can be classified according to their spin.

Bosons $s = 0, 1, 2 \ldots$

spin=0	spin=1
The Higgs boson (pseudo) scalar mesons $\pi^{\pm,0}$, $K^{\pm}$, K^0, $\bar{K}^0, \ldots$ scalar mesons $\sigma, f_0, \ldots$	Carriers of interactions: γ, $W^{\pm}$, Z, g vector mesons $\rho^{\pm,0}$, ω, $\phi, \ldots$ (pseudo) vector mesons $a_1, \ldots$

Fermions $s = 1/2, 3/2, \ldots$

spin=1/2	spin=3/2	spin =5/2
quarks, leptons, neutrinos: q, l, ν_l baryons p, n, Λ, $\Sigma, \ldots$	no elementary particles $\Delta, \ldots, \Omega$, atomic nuclei	no elementary particles baryon resonance, nuclei

Rotation symmetry and angular momentum

What is the spin of any baryon, for example, proton, or pion?
Actually it is its angular momentum $\vec{J}$ built from angular momentum of quarks inside the particle and quarks spins. For example, for any meson consisting of $q\bar{q}$ the spin of the meson is

$$\vec{J} = \vec{S}_1 + \vec{S}_1 + \vec{L} = \frac{\vec{1}}{2} + \frac{\vec{1}}{2} + \vec{L}$$

because the spin of the quark (antiquark) is equal to $1/2$.
If we want to study the mesons with the lowest masses, we take $L = 0$. Then we obtain the spin of the meson

$$\vec{J} = \frac{\vec{1}}{2} + \frac{\vec{1}}{2} \quad \Rightarrow \quad j = 0,\, 1$$

where the general rule of addition of two angular momenta is used

$$\vec{J} = \vec{J}_1 + \vec{J}_2 \quad or \quad j = |j_1 - j_2|,\, |j_1 - j_2| + 1, \ldots, (j_1 + j_2) - 1,\, (j_1 + j_2)$$

Thus for the quark-antiquark system with $L = 0$ we obtain two possibilities

$$J = 0: \quad mesons\,(pseudo)scalar\ octet\ \pi,\ K, \eta, \eta'$$

$$J = 1: \quad mesons\ vector\ octet\ \rho,\ \omega,\ \phi,\ K^*$$

Rotation symmetry and angular momentum

If we add a nonzero orbital moment $L \neq 0$, for example, $L = 1$, then we can construct the meson states with higher spins:

$$\vec{J} = \frac{\vec{1}}{2} + \frac{\vec{1}}{2} + \vec{1} \;\Rightarrow\; J = 0,\, 1,\, 2$$

because the total spin of the quark-antiquark is 0, 1 and we add to this $L = 1$:

$$\vec{J} = \vec{0} + \vec{1} \;\Rightarrow\; j = 1 \qquad \textit{and also } \vec{J} = \vec{1} + \vec{1} \quad\Rightarrow\; j = 0, 1, 2$$

In this way one obtains the so-called tensor mesons with spin=2. Clearly adding $L > 1$, a lot of mesons states with high spin J are generated. They will have the large masses and will be unstable, because can decay to the lower-mass states.

How do we consider baryons consisting of 3 quarks?

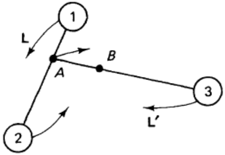
The spin of 3 quarks is obviously

$$\vec{S} = \frac{\vec{1}}{2} + \frac{\vec{1}}{2} + \frac{\vec{1}}{2} = (\frac{\vec{1}}{2} + \frac{\vec{1}}{2}) + \frac{\vec{1}}{2} = (\vec{0},\, \vec{1}) + \frac{\vec{1}}{2} \Rightarrow s = \frac{1}{2},\, \frac{3}{2}$$

However, there can be orbital movement inside the baryon, and there are two independent orbital momenta, $\vec{L}$ and $\vec{L}'$. So that the spin of baryon is

$$\vec{J} = \vec{S} + \vec{L} + \vec{L}'$$

Rotation symmetry and angular momentum



The lowest mass baryons, such as the proton, neutron, Λ, Σ and other baryon octet particles, have $L = L' = 0$. Therefore, we see that the spin of these particles is $\frac{1}{2}$. The other particles, belonging to baryon decuplet, such as $\Delta, \ldots, \Omega$, also have orbital momenta equal to zero, and their spin is $\frac{3}{2}$. The mass of baryon decuplet is larger then the mass of the octet, for example, $m_\Delta > m_p$. We will find out the reason for this in the future.

Flavor symmetry $SU(2)_I$ and isospin

There is an extraordinary observation of Heisenberg in 1932: the neutron is almost identical to the proton. In particular, their masses are close, $m_p = 938.28$ MeV, $m_n = 939.57$ MeV. Heisenberg proposed that they are two 'states' of a single particle – the nucleon. If we forget about the electric charge of the proton, then they would be identical and strong interaction is equal.

$$p = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad n = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad N = \begin{pmatrix} \alpha \\ \beta \end{pmatrix},$$

We can introduce isospin operator $\vec{I}$, similarly to the spin $\vec{S}$ and then the proton has isospin=1/2 and z-projection equal to +1/2, while the neutron has z-projection equal to -1/2. Therefore

$$p = |1/2, +1/2\rangle, \qquad n = |1/2, -1/2\rangle$$

So that they form the **isospin doublet** with the isospin $I = 1/2$.
The main statement is that the strong interactions are invariant with respect to rotation in the internal space, like all interactions are invariant with respect to rotations in our ordinary space.
According to Nother's theorem, $\vec{I}$ is conserved in the strong interaction, like angular momentum $\vec{J}$ is conserved for all interactions due to the rotational symmetry.

Flavor symmetry $SU(2)_I$ and isospin

Presently many isospin multiplets are known, such as isotriplet with $I = 1$

$$\pi^- = |1, -1\rangle, \ \pi^0 = |1, 0\rangle, \ \pi^+ = |1, +1\rangle,$$

isoquadruplet with $I = 3/2$

$$\Delta^- = |3/2, -3/2\rangle, \ \Delta^0 = |3/2, -1/2\rangle, \ \Delta^+ = |3/2, +1/2\rangle, \ \Delta^{++} = |3/2, +3/2\rangle,$$

isosinglet with $I = 0$

$$\Lambda = |0, 0\rangle$$

et cetera for all particles. Apparently the number of particles in the isomultiplet is

$$N = 2I + 1$$

with projections of isospin taking the values $-I$, $-I + 1, \ldots, I - 1$, I. Isospin rotations correspond to going from one particle of multiplet to another, e.g. π^+ should be equivalent to π^0, or π^-.
In fact, some particles in the multiplet have electric charge, the others do not. The electric charge is related to the z-projection of the isospin via the relation of Gell-Mann–Nishijima:

$$Q = I_3 + \frac{1}{2}(A + S)$$

where A is the baryon number, and S is strangeness.

Isospin symmetry $SU(2)_I$ in quark model

Let us assume that the quarks u and d form the isospin doublet, and s quark is the isospin singlet, i.e.

$$u = |1/2, +1/2\rangle, \qquad d = |1/2, -1/2\rangle, \qquad s = |0, 0\rangle$$

Also we know that the baryon number for all quarks is $A = 1/3$, while strangeness of u, d is zero, and s has $S = -1$. Then one can easily check that formula of Gell-Mann–Nishijima is satisfied.

In addition to classification of particles, isospin symmetry has important **dynamical consequences.**
As an example, consider system of two nucleons, each with the isospin $1/2$. This system can have total isospin $\vec{I} = \frac{\vec{1}}{2} + \frac{\vec{1}}{2} \Rightarrow I = 0,\ 1.$

Question: what is the isospin of the deuteron $D = (p\,n)$?

Answer: If it were $I = 1$, then it $d = |1, 0\rangle$ because $I_3 = +1/2$ for proton, $I_3 = -1/2$ for neutron, and for the deuteron $I_3 = 0$.
Because of invariance of the strong interaction under rotation in the isospin space, the states $(pp) = |1, +1\rangle$ and $(nn) = |1, -1\rangle$ would also exist with the same energy! But they do not exist and therefore isospin of deuteron must be zero, i.e $D = |0, 0\rangle$.

Consequences of isospin symmetry

Let us consider 3 different reactions which appear due to the strong interaction:

- (1) $p + p \to D + \pi^+$
- (2) $p + n \to D + \pi^0$
- (3) $n + n \to D + \pi^-$

Can we connect the cross sections of these processes without calculations?
In fact, yes. The final state has isospin equal to 1, i.e. $I_f = 1$ since the deuteron has isospin equal to 0. Then

$$D\,\pi^+ = |1, +1\rangle, \quad D\,\pi^0 = |1, 0\rangle, \quad D\,\pi^- = |1, -1\rangle,$$

What is the isospin of the initial states?
Using the fact that $p = |1/2, +1/2\rangle$ and $n = |1/2, -1/2\rangle$, one can show that

$$|1, +1\rangle = p\,p, \quad |1, -1\rangle = n\,n, \quad |1, 0\rangle = \frac{1}{\sqrt{2}}(p\,n + n\,p)^\dagger$$

and

$$|0, 0\rangle = \frac{1}{\sqrt{2}}(p\,n - n\,p)^\dagger$$

Home task: prove the relations †. For this look in the subject Clebsch-Gordan coefficients for angular momenta, and take $s_1 = 1/2$, $s_2 = 1/2$, and $m_1 = \pm 1/2$, $m_2 = \pm 1/2$.

Consequences of isospin symmetry

Now, the isospin is conserved in the strong-interaction processes, then initial isospin is also 1, i.e. $I_i = 1$. Since we obtained earlier that

$$p\,p = |1, +1\rangle, \quad n\,n = \quad |1, -1\rangle, \quad p\,n = \frac{1}{\sqrt{2}}(|1, 0\rangle + |0, 0\rangle)$$

we see that z-component of isospin is also conserved in all 3 processes (1), (2) and (3). Moreover, in the $p\,n$ state only the component $\frac{1}{\sqrt{2}}|1, 0\rangle$ can contribute, because the other component $\frac{1}{\sqrt{2}}|0, 0\rangle$ does not corresponds to the total isospin 1.
Then we can connect the amplitudes of these 3 reactions:

$$M_1 \,:\, M_2 \,:\, M_3 = 1 \,:\, \frac{1}{\sqrt{2}} \,:\, 1$$

and the cross sections of these reactions, since $\sigma \sim |M|^2$,

$$\sigma_1 \,:\, \sigma_2 \,:\, \sigma_3 = 2 \,:\, 1 \,:\, 2$$

Of course, the reaction with the neutrons $n + n \to D + \pi^-$ is not possible to perform experimentally, because the neutron targets do not exist. The other two reactions are possible, and experiments confirmed the above relation between the cross sections.

Conservation vs. non-conservation of isospin

Let us check what components of isospin are conserved in the strong, electromagnetic and week interactions. For this we use the formula of Gell-Mann-Nishijima (GM-N):

$$Q = I_3 + \frac{1}{2}(A + S)$$

- Strong interaction: baryon number, strangeness, electric charge, isospin I and all its components $\vec{I}$ are conserved.
- Electromagnetic interaction: baryon number, strangeness, electric charge and the component I_3 of the isospin are conserved. However, the other components I_2, I_3 are not conserved. Therefore the total isospin is not conserved. For example, there is the decay $\pi^0 \to \gamma\gamma$ in which $I_i = I_\pi = 1$, but $I_f = 0$ as the leptons and photon do not have isospin.
- Weak interaction: baryon number and electric charge are conserved, but the strangeness is not conserved. Therefore the relation of GM-N can be satisfied only if I_3 is violated, and also $\vec{I}$. For example, there is decay $\Lambda \to p + \pi^-$. Here $I = 0$, $I_3 = 0$ for Λ. In the final state $I = 1/2$, $I_3 = +1/2$ for proton, and $I = 1$, $I_3 = -1$ for pion. The total projection of isospin in the final state is $+1/2 - 1 = -1/2$, and the total isospin in the final state is $\vec{\frac{1}{2}} + \vec{1} \Rightarrow I = 1/2,\, 3/2$. Therefore the total isospin I, and its projection I_3 are not conserved in the week interactions.

Flavor symmetry $SU(3)_F$

As in 1932 the proton and neutron were combined in the pair, it was now increasingly clear that the nucleons, the Λ, the Σ, and the Ξ together can be grouped in the baryon family. They all carry spin 1/2 and their masses are similar.

In fact the masses are different, so that

$$m_N = 940\, MeV, \qquad m_\Xi = 1320\; MeV$$

Nevertheless, it was suggested to regard these eight baryons as a supermultiplet, and this means that they belong in some representation of an enlarged symmetry group, in which the $SU(2)_I$ group of isospin is a subgroup, i.e.

$$SU(2)_I \subset SU(3)_F$$

A problem was related to the fact that the proton and neutron belong to the **fundamental representation** of the group $SU(2)_I$, i.e. there are exactly 2 states, but what particles form the fundamental representation of the group $SU(3)_F$?

The answer was given in the quark model of Gell-Mann: **the 3 quarks** u, d, s **form the basis of the fundamental representation of** $SU(3)_F$. The quarks u, d form the basis of fundamental representation of $SU(2)_I$. The octets of baryons or mesons constitute 8-dimensional representations of $SU(3)_F$, and baryon decuplet – 10-dimensional representation.

Flavor symmetries $SU(4)$, $SU(5)$, . . .

When the charm quark was discovered, the symmetry was promoted to $SU(4)$, and with discovering the bottom quark, to $SU(5)$. However, this was not successful in comparison with observations.

Indeed, let us compare the masses of all particles. For particles in the $SU(2)_I$ isomultiplet:

$$\frac{m_n - m_p}{m_n} \approx 0.001, \quad \frac{m_{\pi^\pm} - m_{\pi^0}}{m_{\pi^0}} \approx 0.003$$

we see that accuracy of symmetry is on the level of 0.1%.

However, if we compare masses of the particles in the supermultiplet of $SU(3)_F$, we see that

$$\frac{m_\Sigma - m_N}{m_\Sigma} \approx 0.2$$

so that the accuracy of $SU(3)_F$ is only 20%.

The situation is even worse for $SU(4)$, because the masses of particles in one supermiltiplet of $SU(4)$ are incomparable, for example

$$m_{\Lambda_c(udc)} \sim 2m_{\Lambda(uds)}$$

Clearly, the case of $SU(5)$ symmetry is even worse, and $SU(6)$ is completely absurd.

Quark masses and flavor symmetries

Why is isospin $SU(2)_I$ a good symmetry, the Eightfold Way $SU(3)_F$ is approximate, and flavor symmetries $SU(4)$, $SU(5)$ and $SU(6)$ are so bad?

Table: Quark masses

Quark flavor	Bare mass (MeV)	Effective (constituent) mass in hadrons (MeV)
u	2	336
d	5	340
s	95	486
c	1300	1550
b	4200	4730
t	174 GeV	177 GeV

We see that the effective mass of the u and d quarks are very similar and are about 350 MeV, so that

$$m_u^{eff} \approx m_d^{eff} \approx 350\, MeV\,,$$

so that in hadrons they are equal. This is the reason for the isospin symmetry $SU(2)_I$. On the other hand, the effective mass of the strange quark $m_s^{eff} \approx 486\, MeV$ is larger than m_u^{eff}, m_d^{eff}, and this causes violation of $SU(3)_F$. The masses of heavy quarks c, b, t are so much different from the masses of the light quarks, that the higher symmetries are not the symmetries at all.

Quark masses and flavor symmetries

This explanation raises two questions.
– 1) Why is effective mass of the quark larger than the bare mass by about 350 MeV?
– 2) Why are the bare masses of the quarks have these values?
In fact, in the Standard Model the masses come from the interaction with the Higgs scalar field. However, the values of the masses are not explained and are just free parameters. Only in theories beyond the Standard Model there is a hope to explain origin and values of the quark (and lepton) masses.

Finally, there maybe a question: why is the neutron mass larger than the proton mass? And not vice verse?

$$M_n \approx (2m_d + m_u) + V_{int}, \qquad M_p \approx (2m_u + m_d) + V_{int} + E_{em}$$

$$M_n - M_p = m_d - m_u - E_{em} \approx 2.1\,MeV - 0.8\,MeV = +1.3\,MeV$$

where E_{em} is the energy, or shift in mass, due to the electric charge of the proton. We can estimate it as for the charged ball of the size of proton, i.e.

$$E_{em} \sim \frac{\alpha}{R_p} \sim 1\,MeV$$

Therefore the difference of masses is due to two effects: i) difference in the mass of the u and d quarks, and ii) electromagnetic mass of the proton.